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Exact solution of the Goryachev–Chaplygin problem in quantum mechanics

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Abstract. The quantum mechanical analogue of the classical integrable system, originally founded by Goryachev and Chaplygin in 1900, is considered in detail. The problem is formulated in terms of the Euclid E(3) group. The Euler–Poisson equations of motions and their integrals are derived. The determination of the spectra of the integrals of motion is equivalent to construction of the special basis of representation of E(3). The eigenvalue problem admits separation in new variables, which are closely connected with two boson creation–annihilation operators. They are the same as in the Majorana representation of the Lorentz group. In the new variables the quantum mechanical equations of motion look similar to the classical ones. The constants of motion are determined by the spectral problems for the two Jacobi-type tridiagonal infinite matrices. Some numerical results are given.

1. Introduction

Since the invention of the inverse scattering method, there has been renewed interest in integrable systems in classical and quantum mechanics. In quantum mechanics it is useful to analyse problems which are solvable in classical mechanics. Separation of the variables in classical mechanics should correspond in quantum mechanics to reduction of the eigenvalue problems for integrals of motion to independent one-dimensional spectral problems. Such a reduction is carried out either by choosing suitable coordinates, if the Hamiltonian is a linear partial differential operator, or by a linear canonical transformation, if the Hamiltonian is of quadratic form in coordinates and momenta. When the Hamiltonian of a system with N degrees of freedom is a more complicated function of coordinates and momenta and admits N integrals of motion in involution, the problem of explicit separation of variables arises. In classical mechanics the problem is solved by the Liouville theorem (Arnol'd 1976). The solution of the corresponding quantum problem is so far unknown. Thus, it is useful to investigate special examples. The well known classical problem of motion of a heavy rigid body about a fixed point can be a good source of suitable examples (Gorr *et al* 1978, Leimanis 1965). Considerable effort was made to find the relations between the parameters involved, assuring existence of the fourth algebraic integral of motion and hence complete integrability of the problem. The celebrated Husson–Poincaré theorem states that at arbitrary initial conditions the problem is integrable in quadratures only in three remarkable cases of Euler, Lagrange and Kovalevskaya. When the initial conditions are chosen in a particular way, new integrable systems appear,

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among which a very interesting case was found by Goryachev and Chaplygin in 1900 (Chaplygin 1948). The Kovalevskaya and Goryachev–Chaplygin (GC) tops in quantum mechanics were discussed by the author (Komarov 1981, 1982). In this paper the solution of the GC top is considered in detail.

The paper is organised as follows. In § 2 the GC problem is formulated in terms of the Euclid group E(3) both in classical and in quantum mechanics. The classical equations of motion are reduced to quadratures. In § 3 the generators of E(3) are represented as functions of two boson creation–annihilation operators. These operators provide the most natural representation of the GC top. Thus, the eigenvalue problem for integrals of motion is reduced to two connected spectral problems for the infinite Jacobi-type matrices and equations of motion are drastically simplified. In § 4 some numerical results are given.

2. The GC top in the terms of E(3)

To fix the position of a rigid body in space one needs three independent parameters. For example, they may be three Euler angles. Often one uses three components of the unit Poisson vector $\boldsymbol{n} = (x_1, x_2, x_3)$ to fix an axis (two parameters) and the angle of rotation about the axis. This parametrisation is very convenient if the external field has axial symmetry and the top rotates freely about the direction of the field.

The Hamiltonian of the GC top in the body-frame can be written as (after appropriate scaling)

$$H = \tfrac{1}{2}(J_1^2 + J_2^2 + 4J_3^2) - bx_1 = \tfrac{1}{2}(J^2 + 3J_3^2) - bx_1 \tag{1}$$

where $\boldsymbol{J} = (J_1, J_2, J_3)$ is the angular momentum and $\boldsymbol{n} = (x_1, x_2, x_3)$ is a unit Poisson vector directed along the uniform field. The principal momenta are in the ratio $1 : 1 : \frac{1}{4}$, thus the top looks like a cucumber. The potential energy $-bx_1$ is a scalar product of the body dipole moment $\boldsymbol{d} = (d, 0, 0)$ directed along its first axis, i.e. across its symmetry axis, and the constant uniform field $\boldsymbol{F} = -\mathcal{F}(x_1, x_2, x_3)$. The parameter b is a product of the moduli of the dipole moment and the field strength. If the field $\boldsymbol{F}$ is gravitational, the dipole moment appears due to non-coincidence of the centre of mass of the top and the fixed point. If the field $\boldsymbol{F}$ is electric, the top rotates about the centre of mass and d is the electric dipole moment of the body. In the last case the Hamiltonian (1) describes the behaviour of a molecule with the momenta inertia ratio $1 : 1 : \frac{1}{4}$ in uniform electric field, if the molecular dipole moment is directed across its symmetry axis. Hamiltonian (1) may also be useful in the theory of magnetism.

In the body-frame the components of the angular momentum J_i and the Poisson vector x_i satisfy the left commutation relations of the Euclid algebra $e(3)$

$$\begin{aligned} &[J_i, J_k] = -\mathrm{i}\varepsilon_{ikl}J_l \qquad && [x_i, x_k] = 0 \\ &[J_i, x_k] = -\mathrm{i}\varepsilon_{ikl}x_l \qquad && i, k = 1, 2, 3 \end{aligned} \tag{2}$$

where the symbol [,] denotes commutator (in quantum mechanics) or the Poisson bracket multiplied by $-\mathrm{i}$ (in classical mechanics). The six Euler–Poisson equations of motion for J_i, x_i are derived with the help of their commutators with the Hamiltonian by the rule

$$\mathrm{d}/\mathrm{d}t = \mathrm{i}[H, \quad]. \tag{3}$$

In classical mechanics one should compute the commutators (3) using the basic algebra (2) only once, then one should suppose that the ordering of the generators is entirely arbitrary.

Direct quantum mechanical calculation gives us the following Euler–Poisson equations

$$\dot{J}_1 = \tfrac{3}{2}\{J_2, J_3\} \qquad \dot{x}_1 = -\tfrac{1}{2}\{J_2, x_3\} + 2\{J_3, x_2\} \tag{4}$$

$$\dot{J}_2 = -\tfrac{3}{2}\{J_1, J_3\} - bx_3 \qquad \dot{x}_2 = \tfrac{1}{2}\{J_1, x_3\} - 2\{J_3, x_1\}$$

$$\dot{J}_3 = bx_2 \qquad \dot{x}_3 = -\tfrac{1}{2}\{J_1, x_2\} + \tfrac{1}{2}\{J_2, x_1\}$$

where $\{\ ,\ \}$ denotes the anticommutator. Equations (4) go over into the classical ones if we do not follow the ordering of the generators in the right-hand sides of equations (4).

In the $e(3)$ Lie algebra there are two Casimir operators

$$r^2 = x_i x_i = 1, \qquad l = x_i J_i = J_i x_i \tag{5}$$

which commute with any generator of E(3) and therefore with the Hamiltonian (1). The trivial integral $r^2 = 1$ in classical mechanics is usually called the integral of cosines. The integral l means conservation of the angular momentum about the direction of the field.

In 1900 Chaplygin (1948) found a remarkable property of the Hamiltonian (1). Namely, its commutator with the function

$$G_{\mathrm{cl}} = 2J_3(J^2 - J_3^2) + 2bx_3 J_1 \tag{6}$$

computed by the rules of classical mechanics, can be written as

$$[H, G_{\mathrm{cl}}]_{\mathrm{cl}} = -2ibl J_2, \tag{7}$$

where l is the projection of the angular momentum onto the field (5). Therefore, if

$$l = 0 \tag{8}$$

the G_{cl} becomes the constant of motion.

Property (7) can be easily established in quantum mechanics as well if one replaces the classical commutators by quantum ones and G_{cl} by G

$$G = 2J_3(J^2 + \tfrac{1}{4} - J_3^2) + b\{x_3, J_1\}. \tag{9}$$

Since l is the Casimir operator, the ordering of l and J_2 in the right-hand side of equation (7) is arbitrary.

In the quantum case, property (7) means that the commutator of the two elements of the universal enveloping algebra of $e(3)$ ($G/2bl$ and H, say) gives us the element J_2 of the $e(3)$ algebra.

The four integrals of motion allow us to integrate the classical problem in quadratures. The results of Chaplygin (1948) are as follows. In the new coordinates u, v

$$u = J_3 + (J^2)^{1/2}, \qquad v = J_3 - (J^2)^{1/2} \tag{10}$$

the equations of motion can be written as

$$\dot{u}(u - v) = (4b^2u^2 - U^2(u))^{1/2} \tag{11}$$

$$\dot{v}(v - u) = (4b^2v^2 - U^2(v))^{1/2}. \tag{12}$$

Here $U(\lambda) = \lambda^3 - 2E\lambda - g$, and E and g are values of the integrals of motion H and

G_{cl}. Chaplygin's method of deriving equations (11), (12) is very specific and cannot be applied in quantum mechanics.

Kozlov (1980) obtained equations (11), (12) by means of the canonical transformation of the Hamiltonian (1), which is first written down in terms of J_3 and $(J^2)^{1/2}$ and their conjugate angles. This way is hardly convenient for the construction of quantum mechanical analogues of equations (11), (12) because the action-angle variables require a very careful approach even for the simplest problems in quantum mechanics.

The equations of motion (11) and (12) are similar to those of the three-body periodic Toda lattice (Flashka and McLaughlin 1976) and coincide with the latter if one replaces $4b^2u^2$, $4b^2u^2 \to 4$ under the square root. Recently Gutzwiller (1981) considered the periodic Toda lattice in quantum mechanics. Common features can be noticed between his analysis and our following discussion. It is likely that a construction of quantum mechanical analogues of the equations (11) and (12) can shed some light on the solution of quantum periodic problems whose classical counterparts are integrable by means of the inverse scattering method.

3. Separation of variables in quantum mechanics

The common eigenfunctions of the operators H, G, $r^2 = 1$, $l = 0$ in quantum mechanics form the new special basis of the irreducible representation of E(3). The operators H, G on the unit sphere S^2: $r^2 = x_i x_i = 1$ are realised as partial differential operators of the second and the third order respectively. There are no coordinates on this sphere which provide a separation of variables. Nevertheless separation of variables can be carried out by a transformation of a more general type.

Let us consider a representation of the $e(3)$ algebra in terms of two boson creation–annihilation operators. This representation is closely connected with the Majorana representation of the Lorentz group (Barut and Rączka 1977). It appears naturally in the quantisation of a rigid rotator by the Dirac procedure as a system subject to constraints (Györgyi and Kövesi-Domokos 1968).

We introduce creation–annihilation operators of two types with the usual boson commutation relations

$$[a_\alpha, a_\beta^+] = \delta_{\alpha\beta}, \qquad [a_\alpha, a_\beta] = 0, \qquad \alpha, \beta = 1, 2 \tag{13}$$

and form bilinear combinations

$$J_i = -\tfrac{1}{2} a_\alpha^+ \sigma_{\alpha\beta}^{(i)} a_\beta, \qquad i = 1, 2, 3 \tag{14a}$$

$$P_i = \tfrac{1}{4}\mathrm{i}(a_\alpha^+ \sigma_{\alpha\beta}^{(i)} (\mathrm{i}\sigma^{(2)})_{\beta\gamma} a_\gamma^+ + a_\alpha (\mathrm{i}\sigma^{(2)})_{\alpha\beta} \sigma_{\beta\gamma}^{(i)} a_\gamma) \tag{14b}$$

$$R_i = \tfrac{1}{4}(a_\alpha^+ \sigma_{\alpha\beta}^{(i)} (-\mathrm{i}\sigma^{(2)})_{\beta\gamma} a_\gamma^+ + a_\alpha (\mathrm{i}\sigma^{(2)})_{\alpha\beta} \sigma_{\beta\gamma}^{(i)} a_\gamma) \tag{14c}$$

$$S = \tfrac{1}{2}(a_\alpha^+ a_\alpha + 1). \tag{14d}$$

Here $\sigma_{\alpha\beta}^{(i)}$ are the matrix elements of the ith Pauli matrix. It is convenient to arrange the ten quantities (14) in the 5×5 skew-symmetric matrix

$$G_{IJ} = \begin{pmatrix} 0 & -J_3 & J_2 & \mathrm{i}P_1 & \mathrm{i}R_1 \\ J_3 & 0 & -J_1 & \mathrm{i}P_2 & \mathrm{i}R_2 \\ -J_2 & J_1 & 0 & \mathrm{i}P_3 & \mathrm{i}R_3 \\ -\mathrm{i}P_1 & -\mathrm{i}P_2 & -\mathrm{i}P_3 & 0 & S \\ -\mathrm{i}R_1 & -\mathrm{i}R_2 & -\mathrm{i}R_3 & -S & 0 \end{pmatrix} \qquad I, J = 1, \ldots, 5. \tag{15}$$

As is known, G_{IJ} obey commutation relations of SO(3, 2)

$$[G_{IJ}, G_{KL}] = \mathrm{i}(\delta_{IL}G_{KJ} + \delta_{IK}G_{JL} + \delta_{JK}G_{LI} + \delta_{JL}G_{IK}) \tag{16}$$

and additional identities

$$\varepsilon_{IPQRS}G_{PQ}G_{RS} = 0 \tag{17a}$$

$$G_{IS}G_{SJ} + G_{JS}G_{SI} = \delta_{IJ}. \tag{17b}$$

Following Györgyi and Kövesi-Domokos (1968) we introduce quantities

$$x_i = S^{-1/2}R_iS^{-1/2}. \tag{18}$$

It is easy to check that x_i and J_i defined by formulae (14*a*) and (18) form the Lie algebra of the E(3) group. Because of the equations (17), in the chosen representation the Casimir operator l is equal to zero and $r^2 = 1$. It should be pointed out that in order to construct the representation of $e(3)$ in terms of a, a^+ it is sufficient to use equations (14*a*, *c*, *d*) and (18) only, without mentioning SO(3, 2).

Now let us write down the Hamiltonian of the GC top in terms of G_{IJ}. Taking into account the definition (18) and the relation $J^2 = S^2 - \frac{1}{4}$, which follows from the property (17*b*), we get the Hamiltonian in the form

$$H = \tfrac{1}{2}(S^2 - \tfrac{1}{4} + 3J_3^2) - bS^{-1/2}R_1S^{-1/2}. \tag{19}$$

Now some manipulations need to be done. Let us pass to the energy representation and multiply equation (19) by $S^{1/2}$ from the left and from the right. The result appears quite natural in new coordinates u, v, which are the counterparts of the classical coordinates (10)

$$\begin{aligned} u &= J_3 + S = \tfrac{1}{2}\{a_2^+, a_2\} > 0 \\ v &= J_3 - S = -\tfrac{1}{2}\{a_1^+, a_1\} < 0. \end{aligned} \tag{20}$$

Using the definition $R_1 = \frac{1}{4}(a_2^{+2} + a_2^2) - (a_1^{+2} + a_1^2)$ we find

$$u^3 - v^3 - 2(E + 1/8)(u - v) - b[(a_2^{+2} + a_2^2) - (a_1^{+2} + a_1^2)] = 0. \tag{21}$$

This obviously admits a separation of variables. We get that the operator $G = G^{(1)}(E, b) = G^{(2)}(E, b)$ is a constant in the energy representation, i.e.

$$G = G^{(2)}(E, b) = u^3 - 2(E + 1/8)u - b(a_2^{+2} + a_2^2) \tag{22a}$$

$$= G^{(1)}(E, b) = v^3 - 2(E + 1/8)v - b(a_1^{+2} + a_1^2). \tag{22b}$$

One can check this by direct computation, excluding E from equations (22) and returning to the original variables J_i, x_i, that the operator G is just the same as in the definition (9).

Equations (22) reduce the eigenvalue problem for H, G to determining the common spectra $g^{(1)}$, $g^{(2)}$ of two one-dimensional Hermitian operators $G^{(1)}$ and $G^{(2)}$. The situation is similar to separation of variables, for example, in the two-Coulomb-centre problem in quantum mechanics (Komarov *et al* 1976). Supposing E to be a free parameter, we first determine the eigenvalues $g_k^{(1)}(E, b)$ and $g_q^{(2)}(E, b)$ depending on their quantum numbers and energy. Then from the equality $g_k^{(1)}(E, b) = g_q^{(2)}(E, b)$ we find the spectra $g_{kq}(b)$, $E_{kq}(b)$.

The equations (22) differ only by sign in the determination of the coordinates (20), therefore the eigenvectors of $G^{(1)}$ are the analytical continuation of those of $G^{(2)}$ on negative $v = -u$. (A similar property exists, for example, for the radial and angular

eigenfunctions of the hydrogen atom in prolate spheroidal coordinates (Komarov *et al* 1976).

The operators $G^{(\alpha)}$ can be represented by the ordinary differential operators on a half-axis of the third (or the sixth) order. It is more convenient to find the eigenfunctions of $G^{(2)}$ ($G^{(1)}$)

$$G^{(2)}|g_q^{(2)}\rangle = g_q^{(2)}|g_q^{(2)}\rangle \tag{23}$$

in the Fock representation

$$|n\rangle = (a^+)^n/(n!)^{1/2}|0\rangle. \tag{24}$$

Supposing $|g^{(2)}\rangle = \Sigma_{n=0}^{\infty} C_n|n\rangle$, we get that the matrix $\langle n|G^{(2)}|n'\rangle$ is tridiagonal and the coefficients C_n obey the three-term recursion relations (TTRR)

$$\begin{aligned} &\gamma_{n-2}C_{n-2} + \beta_n C_n + \gamma_n C_{n+2} = 0 \\ &\gamma_n = -b((n+1)(n+2))^{1/2} \\ &\beta_n = (n+1/2)^3 - 2(E+1/8)(n+1/2) - g^{(2)} \\ &C_{-1} = C_{-2} = 0 \qquad n = 0, 1, \ldots . \end{aligned} \tag{25}$$

It is interesting to note that $\beta_n = U(n+1/2)$, where $U(\lambda)$ is the same cubic form as in the classical equations of motion (11) with the replacement $E \to E + \frac{1}{8}$. The recursion (25) splits into two independent recursions for even and odd n, respectively. Their eigenvalues $g_q^{(2)}$ can be calculated with the help of infinite continuous fractions, which rapidly converge due to the cubic increase of β_n as $n \to \infty$. To analyse qualitatively the spectrum it is useful to apply the quasiclassical approach (Braun 1978), which is based on the analogy between a TTRR and an ordinary differential operator of the second order.

A common eigenvector of H and G is a product of eigenvectors of $G^{(1)}$ and $G^{(2)}$. Multiplying eigenvectors $|g_k^{(1)}\rangle$ and $|g_q^{(2)}\rangle$ of the same parity k and q, we get the solutions, which correspond to integral values of the total angular momentum in the limit $b = 0$. The opposite case makes no physical sense. In fact, we recall that the Casimir operator l is the component of the angular momentum along the field and $l = 0$. It follows immediately that if $l = 0$, only integral values of the total angular momentum can occur in the limit $b = 0$.

From the eigenvectors $|g_k^{(1)}\rangle$ and $|g_q^{(2)}\rangle$ in the Fock representation we can construct the eigenfunctions of H and G on the two-dimensional sphere S^2, using the well known correspondence (Barut and Rączka 1977)

$$\frac{(a_1^+)^{l-m}}{[(l-m)!]^{1/2}} \frac{(a_2^+)^{l+m}}{[(l+m)!]^{1/2}}|0\rangle \leftrightarrow Y_{lm}(\theta, \varphi). \tag{26}$$

Now let us discuss the quantum mechanical equations of motion for u and v. Direct computations with the help of the equations (14), (16) give us

$$\begin{aligned} [H, u] &= -\tfrac{1}{2}bS^{-1/2}(a_2^2 - a_2^{+2})S^{-1/2} \\ [H, v] &= -\tfrac{1}{2}bS^{-1/2}(a_1^2 - a_1^{+2})S^{-1/2}. \end{aligned} \tag{27}$$

If one defines a derivation with respect to a new time

$$\mathrm{d}/\mathrm{d}\tau = 4\mathrm{i}S^{1/2}[H, \quad]S^{1/2} \tag{28}$$

equations of motion for u and v become independent

$$\begin{aligned}(\mathrm{d}/\mathrm{d}\tau)\tfrac{1}{2}\{a_2^+, a_2\} = u_\tau = 2\mathrm{i}b(a_2^{+2} - a_2^2) = \mathrm{i}[G^{(2)}, u] \\ (\mathrm{d}/\mathrm{d}\tau) - \tfrac{1}{2}\{a_1^+, a_1\} = v_\tau = 2ib(a_1^{+2} - a_1^2) = -\mathrm{i}[G^{(1)}, v].\end{aligned} \tag{29}$$

An important property can be readily seen from these equations. Namely, the operators $G^{(\alpha)}$ may be regarded as Hamiltonians with respect to the new time τ.

These equations can be written down in the same form as the classical equations (11), (12). Using the identity

$$(a^{+2} - a^2)^2 = (a^{+2} + a^2)^2 - 4(\tfrac{1}{2}\{a^+, a\})^2 - 3, \tag{30}$$

expressing the sum $a^{+2} + a^2$ from definition (22) and substituting identity (30) into the equations (29), we arrive at

$$\begin{aligned}u_\tau &= 2[4b^2(u^2 + 3/4) - U^2(u, G^{(2)})]^{1/2} \\ v_\tau &= -2[4b^2(v^2 + 3/4) - U^2(v, G^{(1)})]^{1/2}.\end{aligned} \tag{31}$$

Here $U(u, G^{(2)}) = u^3 - 2(E + 1/8)u - G^{(2)}$ is the operator cubic form. The formulae obtained are similar to the classical equations (11), (12).

3. Numerical results

If the field is switched off, i.e. $b = 0$, the eigenstates of the GC top can be labelled by the total angular momentum L and its third component M. These quantum numbers are obviously connected with the quantum numbers of $|g_k^{(1)}\rangle$, $|g_q^{(2)}\rangle$

$$N_\alpha = L + (-1)^\alpha M = \begin{cases} k & \alpha = 1 \\ q & \alpha = 2. \end{cases} \tag{32}$$

The indices L and M play the same role as united atom labels in the two-Coulomb-centre problem (Komarov *et al* 1976). The eigenvectors $|g^{(\alpha)}\rangle$ with the fixed parity can be written as

$$|g^{(\alpha)}\rangle = \sum_{s=0}^{\infty} A_s^{(\alpha)}(-1)^{\alpha s}|n^{(\alpha)}\rangle, \qquad n^{(\alpha)} = 2s + \delta^{(\alpha)}. \tag{33}$$

Here $\delta^{(\alpha)}$ is equal to 0 or 1 depending on the parity of the $L + M$ at $\alpha = 1$ or $L - M$ at $\alpha = 2$, i.e.

$$\delta^{(\alpha)} = \tfrac{1}{2}(1 - (-1)^{N_\alpha}). \tag{34}$$

For the coefficients $A_s^{(\alpha)}$ one gets the TTRR

$$\begin{aligned}&A_{s+1}^{(\alpha)}\gamma_s + A_{s-1}^{(\alpha)}\gamma_{s-1} + A_s^{(\alpha)}\beta_s^{(\alpha)} = 0 \\ &\gamma_s = -b[(n^{(\alpha)} + 1)(n^{(\alpha)} + 2)]^{1/2} \\ &\beta_s^{(\alpha)} = (n^{(\alpha)} + \tfrac{1}{2})^3 - 2(E + \tfrac{1}{8})(n^{(\alpha)} + \tfrac{1}{2}) - (-1)^\alpha g^{(\alpha)} \\ &A_{-1}^{(\alpha)} = 0.\end{aligned} \tag{35}$$

As a consequence, some general properties of spectra g and E can be derived. The corresponding continuous fractions contain only products $\gamma_s\gamma_{s+1}$. Hence the $g(b)$ and $E(b)$ are even functions of b and do not depend on the sign of the field.

If $k = q$ (i.e. $M = 0$), then at a fixed energy E the eigenvalues $g^{(1)}$ and $g^{(2)}$ obviously differ only by sign $g^{(2)} = -g^{(1)}$. So far as the equation (22) implies $g^{(2)} = g^{(1)}$, we get that $g = 0$, if $M = 0$. Similar arguments with the replacement $g_k^{(2)} \leftrightarrow g_q^{(1)}$ show that E is an even function of M, and g is an odd function of M.

The spectra E_{kq} and g_{kq} were calculated by the algorithm developed for the two-Coulomb-centre problem by Truskova (1976). The expansions of $E(b)$, $g(b)$ at small b were used as a good starting approximation. They are readily derived in the first order of the perturbation theory, which is equivalent to the one iteration in the corresponding continuous fraction. Let us denote

$$\theta(L, M) = \frac{(L+M+I)(L+M+2)}{(2L+3)(2L+6M+5)} - \frac{(L+M)(L+M-1)}{(2L-1)(2L+6M-3)}; \tag{36}$$

then we have

$$E(L, M, b) = [L(L+1)+3M^2]/2 - \{b^2/[2(2L+1)]\}[\theta(L, M)+\theta(L-M)] + O(b^4) \tag{37}$$

$$g(L, M, b) = 2M[(L-1/2)^2 - M^2] - [b^2/(2L+1)] \times [(L-M+1/2)\theta(L, M) - (L+M+1/2)\theta(L, -M)] + O(b^4). \tag{38}$$

Some numerical results for a few lowest eigenvalues E and g as a function of b are given in figures 1, 2. The states are labelled by (L, M). Exact crossings occur which are consequences of the separation of variables. For each pair of crossing curves both quantum numbers k and q are simultaneously different, i.e. $k \neq k'$, $q \neq q'$. If quantum numbers for one of the variables (k or q) coincide, the corresponding eigenvalues cannot cross.

There are two patterns of behaviour of $E(b)$. In the first case the energy decreases monotonically while b increases. The GC top is at once aligned along the field. In

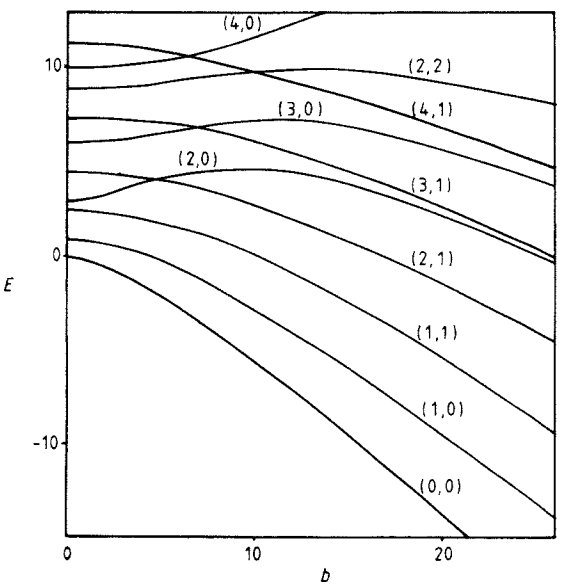

Figure 1. The eigenenergies of the GC top as a function of the field strength. Each curve is labelled by the quantum numbers (L, M) of the free top.

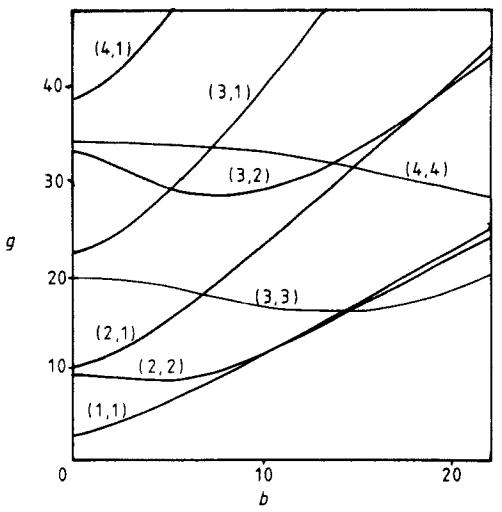

Figure 2. The eigenvalues of the separation constant of the GC top as a function of field strength. Each curve is labelled by the quantum numbers (L, M) of the free top.

the second case the energy has a maximum. The top at first untwists, increasing its energy, and only then draws in the field. To sum up, the energy dependence $E(b)$ for the GC top is similar to those of the usual dipole rotator in a uniform field (Shirley 1963).

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